Prova di ANALISI MATEMATICA 1B 27/6/17

(Corso di laurea in Matematica)

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1. (9 punti) Calcolare

$$\lim_{x\to 0} \frac{\left(\exp\left(\dfrac{1-x}{1+x}\right) - \exp\left(e^{-2x}\right)\right)\sin\left(\dfrac{1-x}{1+x} + e^{-x}\right)}{\sin\left(\dfrac{1-x}{1+x} - e^{-x}\right)\left(\cos(3x) - 1\right)}.$$

2. (6 punti) Risolvere l'equazione in campo complesso

$$ie^{2z} + 2ie^{z} - i - 2\sqrt{3} = 0.$$

3. (8 punti) Calcolare

$$\int_0^{1/3} \frac{\log\left(x + \sqrt{x^2+9}\,\right)}{\sqrt{x^2+9}}\, dx$$

4. (7 punti) Determinare per quali $\cancel{x \in \mathbb{R}}$ $a \in \mathbb{R}^+$ è convergente la serie

$$\int_0^{+\infty} \frac{1-e^{-x}}{(x^a + x^{2a})\left(|\log x|^a + 1\right)}\, dx.$$

(1) $\cos(3x)-1 = 1-\frac{(3x)^2}{2}+o(x^2)-1 = -\frac{9}{2}x^2+o(x^2) \sim -\frac{9}{2}x^2$

$\sin\left(\frac{1-x}{1+x}-e^{-x}\right) \underset{x\to 0}{\sim} \frac{1-x}{1+x}-e^{-x} =$

$= (1-x)(1-x+x^2+o(x^2)) - (1-x+\frac{x^2}{2}+o(x^2))$

$= 1-2x+o(x) - (1-x+o(x^3)) = -x+o(x) \sim -x$

$\sin\left(\frac{1-x}{1+x}+e^{-x}\right) \underset{x\to 0}{\longrightarrow} \sin(2)$

$e^{\frac{1-x}{1+x}} - e^{e^{-2x}} = e^{e^{-2x}}\cdot\left[e^{\frac{1-x}{1+x}-e^{-2x}}-1\right] \underset{x\to 0}{\sim}$

meglio se sviluppo al ~~III~~ ordine, poiché $\mathrm{Den}(x) \underset{x\to 0}{\sim} Ax^3$ $(A\neq 0)$

$e\cdot\left(\frac{1-x}{1+x}-e^{-2x}\right) =$

$= e\cdot\left[(1-x)(1-x+x^2-x^3+o(x^3)) - \left(1-2x+\frac{(2x)^2}{2}-\frac{(2x)^3}{6}+o(x^3)\right)\right]$

$= e\left[\left(1-2x+2x^2+x^3(-1-1)+o(x^3)\right)-\left(1-2x+2x^2-\frac{4}{3}x^3+o(x^3)\right)\right]$

$= e\cdot x^3\left(-2+4/3\right)+o(x^3) \underset{x\to 0}{\sim} -\frac{2e}{3}x^3$

$\Rightarrow f(x) \underset{x\to 0}{\sim} \frac{-\frac{2e}{3}x^3\cdot\sin(2)}{(-x)\cdot\left(-\frac{9}{2}x^2\right)} \underset{x\to 0}{\longrightarrow} -\frac{2}{3}e\sin(2)\frac{2}{9}$

$= \boxed{-\frac{4e\sin(2)}{27}}$

(2) $i e^{2z} + 2i\, e^{z} - i - 2\sqrt{3} = 0$

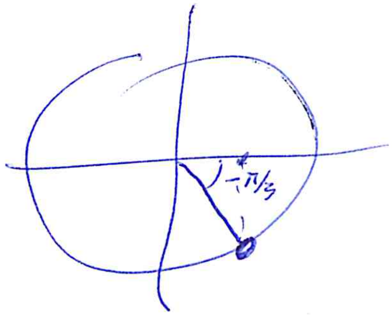
$e^{2z} + 2e^{z} - 1 + 2i\sqrt{3} = 0$

$\parallel$

$e^{2z} + 2e^{z} + 1 - (2 - 2i\sqrt{3})$

$\parallel$

$(e^{z}+1)^2 - (2 - 2i\sqrt{3})$



$\iff (e^{z}+1)^2 = 2 - 2i\sqrt{3} = 4\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= 4\, e^{-i\pi/3}$

$\iff e^{z} + 1 = \pm 2 \cdot e^{-i\pi/6} = \pm 2 \cdot \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \pm(\sqrt{3} - i)$

$e^{z} = -1 \pm \sqrt{3} \mp i$ (a) $e^{z} = -1 + \sqrt{3} - i$

$|-1+\sqrt{3} - i| = \sqrt{(-1+\sqrt{3})^2 + 1} = \sqrt{5 - 2\sqrt{3}}$

Un arg. di $-1+\sqrt{3}-i$ è $\operatorname{arctg}\left(\frac{-1}{\sqrt{3}-1}\right)$

$e^{z} = \sqrt{5 - 2\sqrt{3}} \cdot e^{i \operatorname{arctg}(-1/(\sqrt{3}-1))}$

$z = \frac{1}{2}\log(5 - 2\sqrt{3}) + i\left[\operatorname{arctg}\left(-\frac{1}{\sqrt{3}-1}\right) + 2k\pi\right]$ $k \in \mathbb{Z}$

(b) $e^{z} = -1 - \sqrt{3} + i$: $|(-1-\sqrt{3}) + i| = \sqrt{(-1-\sqrt{3})^2 + 1}$

$= \sqrt{5 + 2\sqrt{3}}$ e un arg. di $-1+\sqrt{3}+i$ è

$\operatorname{arctg}\left(-\frac{1}{\sqrt{3}+1}\right) + \pi$:

$e^{z} = \sqrt{5+2\sqrt{3}} \cdot e^{i\left\{\operatorname{arctg}\left(-\frac{1}{\sqrt{3}+1}\right) + \pi\right\}}$

$\Rightarrow z = \frac{1}{2}\log(5 + 2\sqrt{3}) + i\left[-\frac{1}{\sqrt{3}+1} + \pi + 2k\pi\right]$ $k \in \mathbb{Z}$

(3) $\displaystyle\int_0^{1/3} \frac{\log(x+\sqrt{x^2+9})}{\sqrt{x^2+9}}\,dx = \int_{\sinh^{-1}(0)=0}^{\sinh^{-1}(1/9)} \frac{\log(3\sinh(t)+3\cosh(t))\cdot 3\cosh(t)\,dt}{3\cosh(t)}$

$x = 3\sinh(t)$

$x^2+9 = 9(1+\sinh^2(t)) = 9\cosh^2(t)$

$dx = 3\cosh(t)\,dt$

$t = \sinh^{-1}(x/3)$

$$= \int_0^{\sinh^{-1}(1/9)} \log(6e^t)\,dt = \left[\log(6)t + \frac{t^2}{2}\right]_0^{\sinh^{-1}(1/9)}$$

(4) $x \to 0$ Termini positivi

$$f(x) = \frac{1-(1-x+o(x))}{(x^a+o(x^a))\left[|\log(x)|^a + o(|\log(x)|^a)\right]}$$

$$\sim \frac{x}{x^a\cdot|\log(x)|^a} = \frac{1}{x^{a-1}\cdot|\log(x)|^a}$$

$\int_0^1 f(x)\,dx$ converge se $a-1<1$, cioè se $a<2$.

Quando $a=2$: $\dfrac{1}{x\cdot|\log(x)|^2}$,

quindi $\int_0^1 f(x)\,dx$ conv.

$\int_0^1 f(x)\,dx$ conv. $\Leftrightarrow$ $0<a\leq 2$

$x\to\infty$ $\quad f(x) \sim \dfrac{1}{x^{2a}\cdot|\log x|^a}$

$\int_1^{+\infty} f(x)\,dx$ converge se $2a>1$: $a>1/2$

Se $a = 1/2$, $f(x) \sim \dfrac{1}{x|\log x|^{1/2}}$

e $\int_1^{\infty} f(x)\,dx$ div. $\int_1^{\infty} f(x)\,dx$ conv. $\Leftrightarrow$ $a>1/2$

$\int_0^{+\infty} f(x)\,dx$ converge $\Leftrightarrow$ $\boxed{\frac{1}{2}<a\leq 2}$